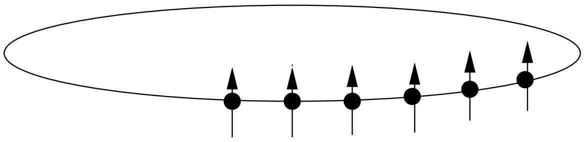
## Lecture VII: Quantum Magnetism and the Ferromagnetic Chain

*Following on from our investigation of the phonon and interacting electron system, we now turn to another example involving bosonic degrees of freedom — the problem of quantum magnetism.*

▷ Spin $S$ Quantum Heisenberg Magnet *spin analogue of discrete harmonic chain*

$$\boxed{\hat{H} = -J \sum_{m=1}^{N} \hat{\mathbf{S}}_m \cdot \hat{\mathbf{S}}_{m+1}}$$

periodic boundary conditions $\hat{\mathbf{S}}_{n+N} = \hat{\mathbf{S}}_n$




*Sign of exchange coupling $J$ depends on material parameters: Coulomb interaction tends to favour ferromagnetism $J > 0$ (cf. Hund's rule) while "superexchange" processes favour antiferromagnetism $J < 0$.*

Aim: To uncover the ground states and nature of low-energy (collective) excitations

▷ Classical ground states?

- Ferromagnet: all spins aligned along a given (arbitrary) direction
  i.e. manifold of continuous degeneracy (cf. crystal)
- Antiferromagnet: (where possible) all neighbouring spins antiparallel — Néel state

▷ Quantum ground states:

$$\hat{H} = -J \sum_m \left[ \hat{S}^z_m \hat{S}^z_{m+1} + \overbrace{\hat{S}^x_m \hat{S}^x_{m+1} + \hat{S}^y_m \hat{S}^y_{m+1}}^{\frac{1}{2}(\hat{S}^+_m \hat{S}^-_{m+1} + \hat{S}^-_m \hat{S}^+_{m+1})} \right]$$

where $\hat{S}^\pm = \hat{S}^x \pm i\hat{S}^y$ denotes spin raising/lowering operator

- Ferromagnet: as classical, e.g. $|\text{g.s.}\rangle = \otimes_{m=1}^N |S^z_m = S\rangle$
  No spin dynamics in $|\text{g.s.}\rangle$, i.e. no zero-point energy! (cf. phonons)
  Manifold of degeneracy explored by acting total spin lowering operator $\sum_m \hat{S}^-_m$
- Antiferromagnet: spin exchange interaction (viz. $\hat{S}^+_m \hat{S}^-_{m+1}$) $\rightsquigarrow$ zero point fluctuations which, depending on dimensionality, may or may not destroy ordered ground state

▷Elementary excitations?

Formation of magnetically ordered state breaks continuous spin rotation symmetry $\rightsquigarrow$ low-energy collective excitations (spin waves or magnons) — cf. phonons in a crystal

Example of general principle known as Goldstone's theorem

However, as with lattice vibrations, 'general theory' is nonlinear
fortunately, low-energy excitations described by free theory

To see this, for large spin $S$, it is helpful to switch to a spin representation in which deviations from $|\text{g.s.}\rangle$ are parameterised as bosons:

$$\begin{array}{ll}|S^z = S\rangle & |n=0\rangle \\ |S^z = S-1\rangle & |n=1\rangle \\ |S^z = S-2\rangle & |n=2\rangle \\ \vdots & \vdots \\ |S^z = -S\rangle & |n=2S\rangle\end{array}$$

i.e. a maximum of $n$ bosons per lattice site ("softcore" constraint)

For ferromagnetic ground state with spins oriented along $z$-axis,
the ground state coincides with the vacuum state $|\text{g.s.}\rangle \equiv |\Omega\rangle$, i.e. $a_m|\Omega\rangle = 0$

Mapping useful when elementary spin wave excitation involves $n \ll 2S$

▷ Mapping of operators: $\hat{S}^z$, $\quad \hat{S}^\pm = \hat{S}^x \pm i\hat{S}^y$?
with $\hbar = 1$, operators obey quantum spin algebra

$$[\hat{S}^\alpha, \hat{S}^\beta] = i\epsilon^{\alpha\beta\gamma}\hat{S}^\gamma \qquad \rightsquigarrow \qquad [\hat{S}^+, \hat{S}^-] = 2\hat{S}^z, \qquad [\hat{S}^z, \hat{S}^\pm] = \pm\hat{S}^\pm$$

cf. bosons: $[a, a^\dagger] = 1, \qquad n = a^\dagger a$

According to mapping, $\hat{S}^z = S - a^\dagger a$;
therefore, to leading order in $S \gg 1$ (spin-wave approximation),

$$\hat{S}^- \simeq (2S)^{1/2} a^\dagger, \qquad \hat{S}^+ \simeq (2S)^{1/2} a$$

In fact, exact equivalence provided by Holstein-Primakoff transformation

$$\hat{S}^- = a^\dagger \left(2S - a^\dagger a\right)^{1/2}, \qquad \hat{S}^+ = (\hat{S}^-)^\dagger, \qquad \hat{S}^z = S - a^\dagger a$$

▷ Applied to <u>FERROMAGNETIC HEISENBERG SPIN $S$ CHAIN</u>, 'spin-wave' approximation:

$$\begin{aligned}\hat{H} &= -J\sum_{m=1}^{N}\left\{\hat{S}^z_m\hat{S}^z_{m+1} + \frac{1}{2}(\hat{S}^+_m\hat{S}^-_{m+1} + \hat{S}^-_m\hat{S}^+_{m+1})\right\} \\ &= -J\sum_m\left\{S^2 - S(a^\dagger_m a_m - a^\dagger_{m+1}a_{m+1}) + S(a_m a^\dagger_{m+1} + a^\dagger_m a_{m+1}) + O(S^0)\right\} \\ &= -J\sum_m\left\{S^2 - 2Sa^\dagger_m a_m + S\left(a^\dagger_m a_{m+1} + \text{h.c.}\right) + O(S^0)\right\}\end{aligned}$$

with p.b.c. $\hat{S}_{m+N} = \hat{S}_m$ and $a_{m+N} = a_m$

To leading order in $S$, Hamiltionian is bilinear in Bose operators;
diagonalised by discrete Fourier transform (exercise)

$$a_k = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{ikn} a_n, \qquad a_n = \frac{1}{\sqrt{N}} \sum_{k}^{\text{B.Z.}} e^{-ikn} a_k, \qquad [a_k, a_{k'}^\dagger] = \delta_{kk'}$$

noting $\sum_n e^{i(k-k')n} = N\delta_{kk'}$

$$\hat{H} = -JNS^2 + \sum_{k}^{\text{B.Z.}} \omega_k a_k^\dagger a_k + O(S^0), \qquad \omega_k = 2JS(1-\cos k) = 4JS\sin^2(k/2)$$

cf. free-particle spectrum

Terms of higher order in $S \rightsquigarrow$ spin-wave interactions