Lecture IV: Second Quantisation

We have seen how the elementary excitations of the quantum chain can be presented in terms of new elementary quasi-particles by the ladder operator formalism. Can this approach be generalised to accommodate other many-body systems? The answer is provided by the method of second quantisation — an essential tool for the development of interacting many-body field theories. The first part of this section is devoted largely to formalism — the second part to applications aimed at developing fluency.

Reference: see Feynman's book on "Statistical Mechanics"

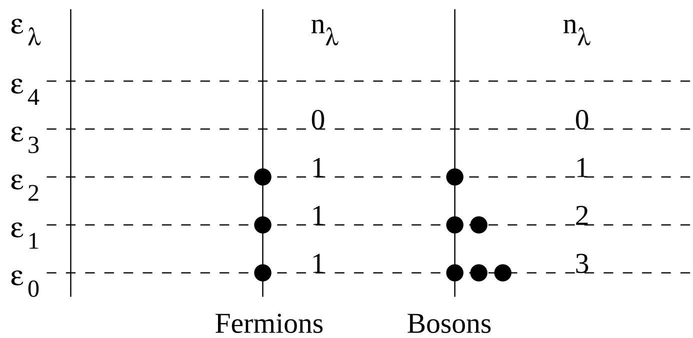
▷ Notations and Definitions

Consider a single-particle Schrodinger equation:

$$\boxed{\hat{H}|\psi_\lambda\rangle = \epsilon_\lambda|\psi_\lambda\rangle}$$

How can one construct a many-body wavefunction?

Particle indistinguishability demands symmetrisation:



e.g. two-particle wavefunction for fermions *i.e. particle 1 in state 1, particle 2...*

$$\psi_F(x_1, x_2) \equiv \frac{1}{\sqrt{2}}(\overbrace{\psi_1}^{\text{state 1}}(\overbrace{x_1}^{\text{particle 1}})\psi_2(x_2) - \psi_2(x_1)\psi_1(x_2))$$

In Dirac notation:

$$|1,2\rangle_F \equiv \frac{1}{\sqrt{2}}\left(|\psi_1\rangle \otimes |\psi_2\rangle - |\psi_2\rangle \otimes |\psi_1\rangle\right)$$

▷ General normalised, symmetrised, N-particle wavefunction of bosons ($\zeta = +1$) or fermions ($\zeta = -1$)

$$\boxed{|\lambda_1, \lambda_2, \ldots \lambda_N\rangle \equiv \frac{1}{\sqrt{N! \prod_{\lambda=0}^{\infty} n_\lambda!}} \sum_{\mathcal{P}} \zeta^{\mathcal{P}} |\psi_{\lambda_{\mathcal{P}1}}\rangle \otimes |\psi_{\lambda_{\mathcal{P}2}}\rangle \ldots \otimes |\psi_{\lambda_{\mathcal{P}N}}\rangle}$$

- n_λ — no. of particles in state λ
  (for fermions, Pauli exclusion: $n_\lambda = 0, 1$, i.e. $|\lambda_1, \lambda_2, \ldots \lambda_N\rangle$ is a Slater determinant)

- $\sum_{\mathcal{P}}$: Summation over $N!$ permutations of $\{\lambda_1, \dots \lambda_N\}$
  required by particle indistinguishability
- Parity $\mathcal{P}$ — no. of transpositions of two elements which brings permutation
  $(\mathcal{P}_1, \mathcal{P}_2, \cdots \mathcal{P}_N)$ back to ordered sequence $(1, 2, \cdots N)$

Evidently, "first quantised" representation looks clumsy!
motivates alternative representation...

▷ SECOND QUANTISATION

Define vacuum state: $|\Omega\rangle$, and set of field operators a_λ and adjoints $a_\lambda^\dagger$ — no hats!

$$\boxed{a_\lambda|\Omega\rangle = 0, \qquad \frac{1}{\sqrt{\prod_{\lambda=0}^{\infty} n_\lambda!}} \prod_{i=1}^{N} a_{\lambda_i}^\dagger |\Omega\rangle = |\lambda_1, \lambda_2, \dots \lambda_N\rangle}$$

cf. bosonic ladder operators for phonons *N.B. ambiguity of ordering?*

Field operators fulfil commutation relations for bosons (fermions)

$$\left[a_\lambda, a_\mu^\dagger\right]_{-\zeta} = \delta_{\lambda\mu}, \qquad \left[a_\lambda, a_\mu\right]_{-\zeta} = \left[a_\lambda^\dagger, a_\mu^\dagger\right]_{-\zeta} = 0$$

where $[\hat{A}, \hat{B}]_{-\zeta} \equiv \hat{A}\hat{B} - \zeta\hat{B}\hat{A}$ is the commutator (anti-commutator)

- Operator $a_\lambda^\dagger$ creates particle in state λ, and a_λ annihilates it
- Commutation relations imply Pauli exclusion for fermions: $a_\lambda^\dagger a_\lambda^\dagger = 0$
- Any N-particle wavefunction can be generated by application of set of
  N operators to a unique vacuum state

  e.g. $\quad |1, 2\rangle = a_2^\dagger a_1^\dagger |\Omega\rangle$

- Symmetry of wavefunction under particle interchange maintained by
  commutation relations of field operators

  e.g. $\quad |1, 2\rangle = a_2^\dagger a_1^\dagger |\Omega\rangle = \zeta a_1^\dagger a_2^\dagger |\Omega\rangle$

  (So, providing one maintains a consistent ordering convention,
  the nature of that convention doesn't matter)

▷ Fock space: Defining $\mathcal{F}_N$ to be 'linear span' of all N-particle states $|\lambda_1, \lambda_2, \cdots \lambda_N\rangle$
Fock space $\mathcal{F}$ is defined as 'direct sum' $\oplus_{N=0}^{\infty} \mathcal{F}_N$

- General state $|\phi\rangle$ of the Fock space is linear combination of states
  with any number of particles
- Note that the vacuum state $|\Omega\rangle$ (sometimes written as $|0\rangle$) is distinct from zero!

$$\begin{tikzcd} \cdots \arrow[r, bend left] & \boxed{\mathcal{F}_2} \arrow[l, bend left] \arrow[r, bend left, "a"] & \boxed{\mathcal{F}_1} \arrow[l, bend left, "a^+"] \arrow[r, bend left, "a"] & \boxed{\mathcal{F}_0} \arrow[l, bend left, "a^+"] \arrow[r, bend left, "a"] & 0 \end{tikzcd}$$

▷ <u>Change of basis:</u>

Using the resolution of identity $\mathbf{1} \equiv \sum_\lambda |\lambda\rangle\langle\lambda|$, we have $\overbrace{|\tilde{\lambda}\rangle}^{a^\dagger_{\tilde{\lambda}}|\Omega\rangle} = \sum_\lambda \overbrace{|\lambda\rangle}^{a^\dagger_\lambda|\Omega\rangle} \langle\lambda|\tilde{\lambda}\rangle$

$$\text{i.e.} \quad a^\dagger_{\tilde{\lambda}} = \sum_\lambda \langle\lambda|\tilde{\lambda}\rangle a^\dagger_\lambda, \qquad \text{and} \quad a_{\tilde{\lambda}} = \sum_\lambda \langle\tilde{\lambda}|\lambda\rangle a_\lambda$$

E.g. Fourier representation: $a_\lambda \equiv a_k$, $a_{\tilde{\lambda}} \equiv a(x)$

$$a(x) = \sum_k \overbrace{\langle x|k\rangle}^{e^{ikx}/\sqrt{L}} a_k, \qquad a_k = \frac{1}{\sqrt{L}} \int_0^L dx\, e^{-ikx} a(x)$$

▷ <u>Occupation number operator:</u> $\hat{n}_\lambda = a^\dagger_\lambda a_\lambda$ measures no. of particles in state λ

e.g. (bosons)

$$a^\dagger_\lambda a_\lambda (a^\dagger_\lambda)^n|\Omega\rangle = a^\dagger_\lambda \overbrace{a_\lambda a^\dagger_\lambda}^{1+a^\dagger_\lambda a_\lambda} (a^\dagger_\lambda)^{n-1}|\Omega\rangle = (a^\dagger_\lambda)^n|\Omega\rangle + (a^\dagger_\lambda)^2 a_\lambda (a^\dagger_\lambda)^{n-1}|\Omega\rangle = \cdots = n(a^\dagger_\lambda)^n|\Omega\rangle$$

Exercise: check for fermions