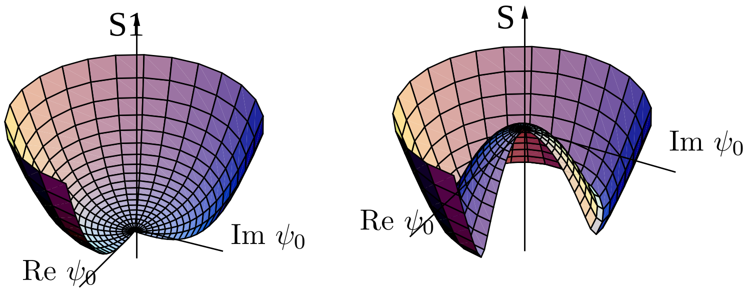
Lecture XX: Superfluidity

Previously, we have seen that, when treated in a mean-field or saddle-point approximation, the field theory of the weakly interacting Bose gas shows a transition to a Bose-Einstein condensed phase when $\mu = 0$ where the order parameter, the complex condensate wave-function ψ_0 acquires a non-zero expectation value, $|\psi_0| = \gamma \equiv \sqrt{\mu L^d/g}$. The spontaneous breaking of the continuous symmetry associated with the phase of the order parameter is accompanied by the appearance of massless collective phase fluctuations. In the following, we will explore the properties of these fluctuations and their role in the phenomenon of superfluidity.

▷ Starting with model action for the Bose system, ($\hbar = 1$)

$$S[\bar\psi, \psi] = \int_0^\beta d\tau \int d^d r \left[\bar\psi(\mathbf{r},\tau)\left(\partial_\tau - \frac{\partial^2}{2m} - \mu\right)\psi(\mathbf{r},\tau) + \frac{g}{2}(\bar\psi(\mathbf{r},\tau)\psi(\mathbf{r},\tau))^2\right]$$

a saddle-point analysis of the action revealed that, for $\mu > 0$, the field ψ acquires a constant non-zero expectation value: $\bar\psi_0 = \psi_0 = (\mu L^d/g)^{1/2} \equiv \gamma$



In the following, we will explore the effect of fluctuations around the mean-field

To do so, it is convenient to effect the reparameterisation $\psi(\mathbf{r},\tau) = [\rho(\mathbf{r},\tau)]^{1/2} e^{i\phi(\mathbf{r},\tau)}$

Using

1. $$\int_0^\beta d\tau \bar\psi \partial_\tau \psi = \overbrace{\int_0^\beta d\tau \rho^{1/2}\partial_\tau \rho^{1/2}}^{\frac{1}{2}\int_0^\beta d\tau \partial_\tau(\rho^{1/2}\rho^{1/2}) = -\frac{\rho}{2}\Big|_0^\beta = 0} + \int_0^\beta d\tau i\rho\partial_\tau\phi$$
2. $$\partial(\rho^{1/2}e^{i\phi}) = \left(\frac{1}{2\rho^{1/2}}\partial\rho + i\rho^{1/2}\partial\phi\right)e^{i\phi}$$
3. $$\int_0^\beta d\tau \bar\psi \partial^2\psi = -\int_0^\beta d\tau \partial\bar\psi \cdot \partial\psi = -\int_0^\beta \left(\frac{1}{4\rho}(\partial\rho)^2 + \rho(\partial\phi)^2\right)$$

the action takes the form

$$S[\rho, \phi] = \int_0^\beta d\tau \int d^d r \left\{ i\rho\partial_\tau\phi + \frac{1}{2m}\left[\frac{1}{4\rho}(\partial\rho)^2 + \rho(\partial\phi)^2\right] - \mu\rho + \frac{g\rho^2}{2}\right\}$$

Then, discarding gradient terms involving massive fluctuations $\delta\rho$,
an expansion in $\delta\rho \equiv \rho - \rho_0$ at $\mu = \mu_{\rm BEC} = 0$

$$S[\delta\rho, \phi] \simeq S_0[\rho_0] + \int_0^\beta d\tau \int d^d r \left[i\delta\rho\partial_\tau\phi + \frac{g\delta\rho^2}{2} + \frac{\rho_0}{2m}(\partial\phi)^2 \right]$$

- First term has canonical structure 'momentum $\times\, \partial_\tau$ (coordinate)'
  cf. canonically conjugate pair
- Second term records energy cost of "massive" fluctuations from
  Mexican hat potential minimum
- Third term measures energy cost of spatially varying massless phase flucutations:
  i.e. ϕ is a Goldstone mode

Gaussian integration over $\delta\rho$:

$$\int D(\delta\rho) \exp\Big[-\int_0^\beta d\tau \int d^d r \overbrace{\left(i\delta\rho\partial_\tau\phi + \frac{g\delta\rho^2}{2} \right)}^{\frac{g}{2}\left(\delta\rho + \frac{i}{g}\partial_\tau\phi\right)^2 + \frac{(\partial_\tau\phi)^2}{2g}} \Big] = \text{const.} \times \exp\left[-\int_0^\beta d\tau \int d^d r \frac{(\partial_\tau\phi)^2}{2g} \right]$$

$\rightsquigarrow$ effective low energy action

$$S[\phi] \simeq S_0 + \frac{1}{2}\int_0^\beta d\tau \int d^d r \left[\frac{1}{g}(\partial_\tau\phi)^2 + \frac{\rho_0}{m}(\partial\phi)^2 \right].$$

cf. Lagrangian formulation of harmonic medium (or massless Klein-Gordon field)

$$S = \int dt \int d^d r \left[\frac{m}{2}\dot{\phi}^2 - \frac{1}{2}k_s a^2 (\partial\phi)^2 \right] = \int dx\, \partial^\mu\phi\partial_\mu\phi$$

i.e. low-energy excitations involve collective phase fluctuations with a spectrum $\omega_{\mathbf{k}} = \frac{g\rho_0}{m}|\mathbf{k}|$

▷ <u>Physical ramifications:</u> consider quantum mechanical current density operator

$$\begin{aligned}\hat{\mathbf{j}}(\mathbf{r},\tau) &= \frac{1}{2}\left[a^\dagger(\mathbf{r},\tau)\frac{\hat{\mathbf{p}}}{m}a(\mathbf{r},\tau) - \left(\frac{\hat{\mathbf{p}}}{m}a^\dagger(\mathbf{r},\tau)\right)a(\mathbf{r},\tau) \right] \\ &\xrightarrow{\text{fun. int}} \frac{i}{2m}\left[(\partial\bar{\psi}(\mathbf{r},\tau))\psi(\mathbf{r},\tau) - \bar{\psi}(\mathbf{r},\tau)\partial\psi(\mathbf{r},\tau) \right] \simeq \frac{\rho_0}{m}\partial\phi(\mathbf{r},\tau)\end{aligned}$$

i.e. $\partial\phi$ is measure of (super)current flow

Variation of action $S[\delta, \phi] \rightsquigarrow$

$$i\partial_\tau\phi = -g\delta\rho, \qquad i\partial_\tau\delta\rho = \frac{\rho_0}{m}\partial^2\phi = \partial\cdot\mathbf{j}$$

- First equation: system adjusts to spatial fluctuations of density
  by dynamical phase fluctuation

- Second equation $\rightsquigarrow$ continuity equation (conservation of mass)

Crucially, stationary equations possess steady state solution with non-vanishing current flow: setting $\partial_\tau \phi = \partial_\tau \delta\rho = 0$, obtain $\delta\rho = 0$ and $\partial \cdot \mathbf{j} = 0$

i.e. for $T < T_{\rm BEC}$, a configuration with a uniform density profile can support a steady state divergenceless (super)flow

Notice that a 'mass term' in the phase ϕ action would spoil this property, i.e. the phenomenon of superflow is intimately linked to the Goldstone mode

$\triangleright$ Steady state current flow in normal environments is prevented by the mechanism of energy dissipation, i.e. particles scatter off imperfections inside the system and thereby converting part of their energy into the creation of elementary excitations

How can dissipative loss of energy be avoided?

Trivially, no energy can be exchanged if there are no elementary excitations to create

In reality, this means that the excitations of the system should be energetically inaccessible (k.e. of carriers too small to create excitations)

But this is not the case here! there is no energy gap ($\omega_{\mathbf{k}} \propto |\mathbf{k}|$)

However, there is an ingenuous argument due to Landau (see notes) showing that a linear excitation spectrum can stabilize dissipationless transport