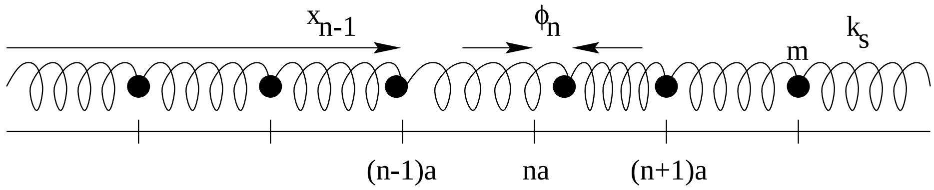
Lecture II: Collective Excitations: From Particles to Fields

Free Scalar Field Theory: Phonons

The aim of this course is to develop the machinery to explore the properties of quantum systems with very large or infinite numbers of degrees of freedom. To represent such systems it is convenient to abandon the language of individual elementary particles and speak about quantum fields. In this lecture, we will consider the simplest physical example of a free or non-interacting many-particle theory theory which will exemplify the language of classical and quantum fields. Our starting point is a toy model of a mechanical system describing a classical chain of atoms coupled by springs.

▷ Discrete elastic chain



Equilibrium position $\bar{x}_n \equiv na$; natural length a; spring constant k_s

Goal: to construct and quantise a classical field theory
for the collective (longitundinal) vibrational modes of the chain

▷ Discrete Classical Lagrangian:

$$L = T - V = \sum_{n=1}^{N} \Big(\overbrace{\frac{m}{2}\dot{x}_n^2}^{\text{k.e.}} - \overbrace{\frac{k_s}{2}(x_{n+1} - x_n - a)^2}^{\text{p.e. in spring}} \Big)$$

assume periodic boundary conditions (p.b.c.) $x_{N+1} = Na + x_1$ (and set $\dot{x}_n \equiv \partial_t x_n$)

Using displacement from equilibrium $\phi_n = x_n - \bar{x}_n$

$$L = \sum_{n=1}^{N} \left(\frac{m}{2}\dot{\phi}_n^2 - \frac{k_s}{2}\left(\phi_{n+1} - \phi_n\right)^2 \right), \qquad \text{p.b.c}: \quad \phi_{N+1} \equiv \phi_1$$

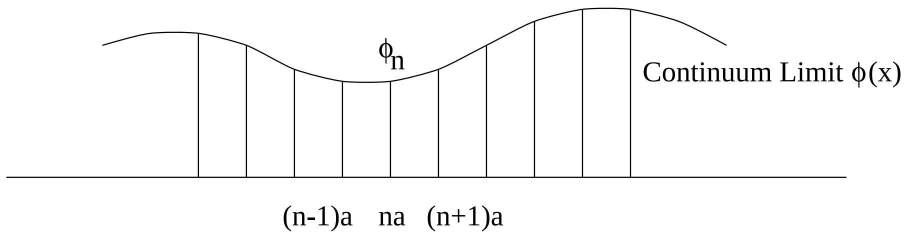
In principle, one can obtain exact solution of discrete equation of motion — see PS I

However, typically, one is not concerned with behaviour on 'atomic' scales:

1. for such purposes, modelling is too primitive!
   anharmonic contributions
2. such properties are in any case 'non-universal'

Aim here is to describe low-energy collective behaviour — generic, i.e. <u>universal</u>

In this case, it is often permissible to neglect the discreteness of the microscopic entities of the system and to describe it in terms of effective continuum degrees of freedom.



▷ Continuum Lagrangian

Describe ϕ_n as a smooth function $\phi(x)$ of a continuous variable x;
makes sense if $\phi_{n+1} - \phi_n \ll a$ (i.e. gradients small)

$$\phi_n \to a^{1/2}\phi(x)\Big|_{x=na}, \qquad \phi_{n+1} - \phi_n \to a^{3/2}\partial_x\phi(x)\Big|_{x=na}, \qquad \sum_n \longrightarrow \frac{1}{a}\int_0^{L=Na} dx$$

N.B. $[\phi(x)] = L^{1/2}$

$$\overbrace{L[\phi] = \int_0^L dx\ \mathcal{L}(\phi, \partial_x\phi, \dot{\phi}),}^{\text{Lagrangian functional}} \qquad \overbrace{\mathcal{L}(\phi, \partial_x\phi, \dot{\phi}) = \frac{m}{2}\dot{\phi}^2 - \frac{k_s a^2}{2}(\partial_x\phi)^2}^{\text{Lagrangian density}}$$

N.B. $[\mathcal{L}]$ = [energy]/[length]

▷ Classical action

$$\boxed{S[\phi] = \int dt\ L[\phi] = \int dt \int_0^L dx\ \mathcal{L}(\phi, \partial_x\phi, \dot{\phi})}$$

- N-point particle degrees of freedom $\mapsto$ continuous classical field $\phi(x)$
- Dynamics of $\phi(x)$ specified by functionals $L[\phi]$ and $S[\phi]$

What are the corresponding equations of motion...?

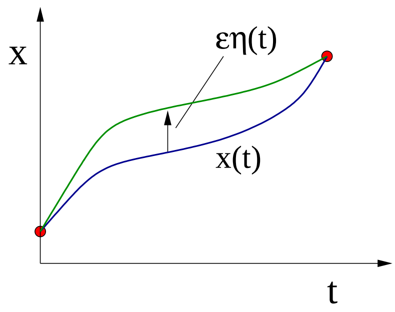
▷ Hamilton's Extremal Principle (HEP): (Revision)

Suppose classical point particle $x(t)$ described by action $S[x] = \int dt\ L(x, \dot{x})$

HEP: configurations $x(t)$ that are realised are those that extremise the action
i.e. for any smooth function $\eta(t)$, the "variation"

$$\delta S[x] \equiv \lim_{\epsilon\to 0}\frac{1}{\epsilon}(S[x + \epsilon\eta] - S[x]) = 0$$

to 1st order in ϵ, action must by stationary



Extremal condition $\mapsto$ Euler-Lagrange equations of motion

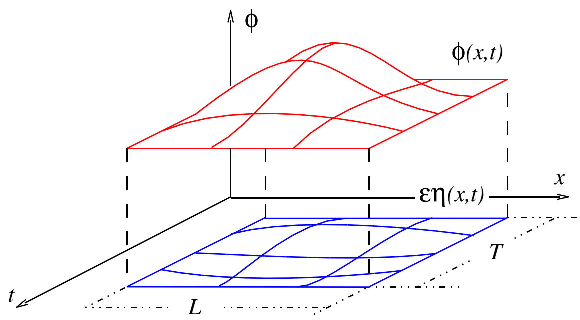
$$S[x+\epsilon\eta] = \int_0^t dt\, L(x+\epsilon\eta, \dot{x}+\epsilon\dot{\eta}) = \int_0^t dt\,(L(x,\dot{x}) + \epsilon\eta\partial_x L + \epsilon\dot{\eta}\partial_{\dot{x}} L) + O(\epsilon^2)$$

$$\delta S[x] = \int dt\,(\eta\partial_x L + \dot{\eta}\partial_{\dot{x}} L) \stackrel{\text{by parts}}{=} \int dt\, \overbrace{\left(\partial_x L - \frac{d}{dt}(\partial_{\dot{x}} L)\right)}^{=0} \eta = 0$$

Note: boundary terms vanish by construction

(The variation $\dfrac{\delta L}{\delta x} = \partial_x L - \dfrac{d}{dt}(\partial_{\dot{x}} L)$ is an example of functional differentiation.
A formal discussion of its legitimacy is included in the lecture notes.)

How does HEP generalise to continuum field $x \mapsto \phi(x)$?



Apply same extremal principle: $\phi(x,t) \mapsto \phi(x,t) + \epsilon\eta(x,t)$
with both ϕ and η both periodic in x

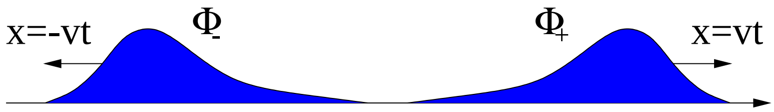
$$S[\phi+\epsilon\eta] = S[\phi] + \epsilon\int_0^t dt \int_0^L dx\left(m\dot{\phi}\dot{\eta} - k_s a^2 \partial_x\phi\partial_x\eta\right) + O(\epsilon^2).$$

Integrating by parts *boundary terms vanish by construction:* $\eta\dot{\phi}|_0^t = 0 = \eta\partial_x\phi|_0^L$

$$\delta S = -\int_0^t dt \int_0^L dx\left(m\ddot{\phi} - k_s a^2\partial_x^2\phi\right)\eta = 0$$

Since $\eta(x,t)$ is an arbitrary smooth function, we must have

$$\left(m\partial_t^2 - k_s a^2\partial_x^2\right)\phi = 0$$



i.e. $\phi(x,t)$ obeys classical wave equation

General solutions are of the form: $\phi_+(x+vt)+\phi_-(x-vt)$
where $v = a\sqrt{k_s/m}$ is sound wave velocity and $\phi_\pm$ are arbitrary smooth functions

More generally, for the Lagrangian density $\mathcal{L} = \mathcal{L}(\phi, \partial_x\phi, \dot{\phi})$,

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial(\partial_x\phi)} = 0$$

▷ <u>Comments</u>

- Low-energy <u>collective</u> excitations — known as phonons — are lattice vibrations propagating as sound waves to left or right at constant velocity v

- Trivial behaviour of model is a direct consequence of its simplistic definition:

  Lagrangian is quadratic in fields $\mapsto$ linear equation of motion

  Higher order couplings (i.e. interactions) $\mapsto$ dissipation and dispersion

- L is said to be a 'free (i.e. non-interacting) scalar (i.e. one-component) field theory'

- In higher dimensions, field has vector components $\mapsto$ transverse and longintudinal modes

Variational principle is example of <u>functional analysis</u> — a useful (but not essential concept for this course) — see lecture notes