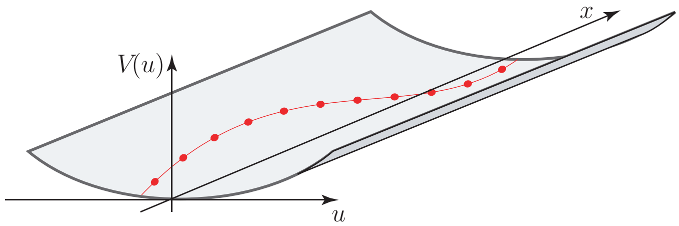
Lecture XI: Statistical Mechanics and Semi-Classics

▷ <u>Connection of Path Integral to Classical Statistical Mechanics</u>

Consider flexible string held under constant tension and confined to 'gutter' potential



Potential energy stored in spring due to line tension:

$$\text{from segment } x \text{ to } x+dx,\ dV_T = T\,\overbrace{[(dx^2+du^2)^{1/2}-dx]}^{\text{extension}} \simeq T\,dx\,(\partial_x u)^2/2$$

$$V_T[\partial_x u] \equiv \int dV_T = \frac{1}{2}\int_0^L dx\ T\,(\partial_x u(x))^2$$

External (gutter) potential: $V_{\text{ext}}[u] \equiv \int_0^L dx\ V[u(x)]$

According to Boltzmann principle, equilibrium partition function

$$\boxed{\mathcal{Z} = \operatorname{tr}\left(e^{-\beta F}\right) = \int Du(x)\ \exp\left[-\beta\int_0^L dx\left(\frac{T}{2}(\partial_x u)^2 + V(u)\right)\right]}$$

cf. quantum mechanical transmission amplitude

▷ <u>Mapping</u>:

$$\mathcal{Z} = \int_{\text{b.c.}} Dq(t)\ \exp\left[\frac{i}{\hbar}\int_0^t dt'\left(\frac{m\dot{q}^2}{2} - V(q)\right)\right]$$

Wick rotation $t \to -i\tau \mapsto$ imaginary (Euclidean) time path integral

$$\int_0^t idt'\ (\partial_{t'}q)^2 \longrightarrow -\int_0^\tau d\tau'(\partial_{\tau'}q)^2, \qquad -\int_0^t idt' V(q) \longrightarrow -\int_0^\tau d\tau' V(q)$$

$$\mathcal{Z} = \int_{\text{b.c.}} Dq\,\exp\left[-\frac{1}{\hbar}\int_0^\tau d\tau'\left(\frac{m}{2}(\partial_{\tau'}q)^2 + V(q)\right)\right] \qquad \text{N.B. change of relative sign!}$$

(a) <u>Classical partition function</u> of one-dimensional system
coincides with quantum mechanical amplitude

$$\mathcal{Z} = \int dq\ \langle q|e^{-i\hat{H}t/\hbar}|q\rangle\Big|_{t=-i\tau}$$

where time is imaginary, and $\hbar$ plays the role of temperature

More generally, path integral for d-dimensional quantum system
corresponds to classical statistical mechanics of $d+1$-dimensional system

(b) Quantum partition function

$$\mathcal{Z} = \mathrm{tr}(e^{-\beta\hat{H}}) = \int dq\, \langle q|e^{-\beta\hat{H}}|q\rangle$$

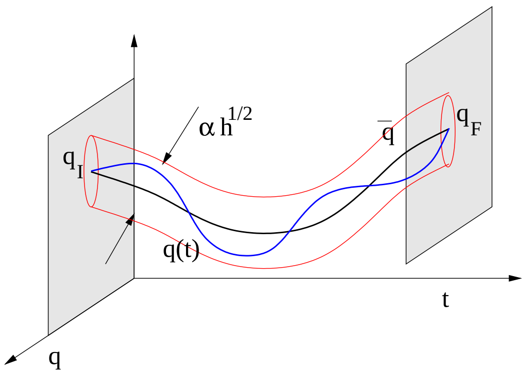
i.e. $\mathcal{Z}$ can be interpreted through dynamical transition amplitude $\langle q|e^{-i\hat{H}t/\hbar}|q\rangle$
evaluated at imaginary time $t = -i\hbar\beta$.

(c) In semi-classical limit ($\hbar \to 0$), PI dominated by stationary configurations of action
$S[p,q] = \int dt(p\dot{q} - H(p,q))$

$$\begin{aligned}\delta S &= S[p+\delta p, q+\delta q] - S[p,q] \\ &= \int dt\, [\delta p\dot{q} + p\delta\dot{q} - \delta p\partial_p H - \delta q\partial_q H] + O(\delta p^2, \delta q^2, \delta p\delta q) \\ &= \int dt\, [\delta p\,(\dot{q} - \partial_p H) + \delta q\,(-\dot{p} - \partial_q H)] + O(\delta p^2, \delta q^2, \delta p\delta q)\end{aligned}$$

i.e. Hamilton's classical e.o.m.: $\dot{q} = \partial_p H$, $\dot{p} = -\partial_q H$ with b.c. $q(0) = q_I$, $q(t) = q_F$

(Similarly, with Lagrangian formulation : $\delta S = 0 \Rightarrow (d_t\partial_{\dot{q}} - \partial_q)\, L(q,\dot{q}) = 0$)



Contributions to PI from fluctuations around classical paths?

Usually, exact evaluation of PI impossible — resort to approximation schemes...

▷ Saddle-point and Stationary Phase analysis

Consider integral over single variable

$$I = \int_{-\infty}^{\infty} dz\, e^{-f(z)}$$

Integral dominated by minima of $f(z)$; suppose unique i.e. $f'(z_0) = 0$

Taylor expand around minimum: $f(z) = f(z_0) + (z - z_0)\overbrace{f'(z_0)}^{\mapsto 0} + \frac{1}{2}(z - z_0)^2 f''(z_0) + \cdots$

$$I \simeq e^{-f(z_0)} \int_{-\infty}^{\infty} dz\, e^{-(z-z_0)^2 f''(z_0)/2} = \sqrt{\frac{2\pi}{f''(z_0)}} e^{-f(z_0)}$$

$$\text{Example}: \quad \Gamma(s+1) = \int_0^\infty dz z^s e^{-z} = \int_0^\infty dz\, e^{-f(z)}, \qquad f(z) = z - s \ln z$$

$f'(z) = 1 - s/z$ i.e. $z_0 = s$, $f''(z_0) = s/z_0^2 = 1/s$

i.e. $\Gamma(s+1) \simeq \sqrt{2\pi s} e^{-(s - s\ln s)}$ — Stirling's formula

If minima not on contour of integration — deform contour through saddle-point

e.g. $\Gamma(s+1)$, s complex

What if exponent complex? Fast phase fluctuations $\leadsto$ cancellation

i.e. expand around region of slowest (i.e. stationary) phase and use identity

$$\int_{-\infty}^{\infty} dz\, e^{iaz^2/2} = \sqrt{\frac{2\pi}{a}} e^{i\pi/4}$$

▷ Can we apply same approach to analyse the FPI?

Yes: but we we must develop new technology;

basic tool of QFT — the Gaussian functional integral!