Lecture VII: Luttinger Liquid Theory[4]

▷ References:

Original papers:

- Sin-itiro Tomonaga, *Remarks on Bloch's Method of Sound Waves applied to Many-Fermion Problems*, Progress of Theoretical Physics **5**, 544 (1950).
- J. M. Luttinger, Journal of Mathematical Physics **4**, 1154 (1963).
- F. D. M. Haldane, *'Luttinger liquid theory' of one-dimensional quantum fluids:I. Properties of the Luttinger model and their extension to the general 1D interacting spinless Fermi gas*, Journal of Physics C: Solid State Physics **14**, 2585 (1981).

Books and Reviews:

- A. O. Gogolin, A. A. Nersesyan, and A. M. Tsvelik, *Bosonization and Strongly Correlated Systems* (Cambridge University Press, 1998).
- H. J. Schulz, G. Cuniberti, and P. Pieri, *Fermi Liquids and Luttinger Liquids*, `cond-mat/9807366`.

▷ Why is 1D special?

In 1D, particle-hole excitations with a given momentum have the same kinetic energy. This holds even in case of interacting electrons. The only requirement is that the spectrum is linear about the Fermi points.

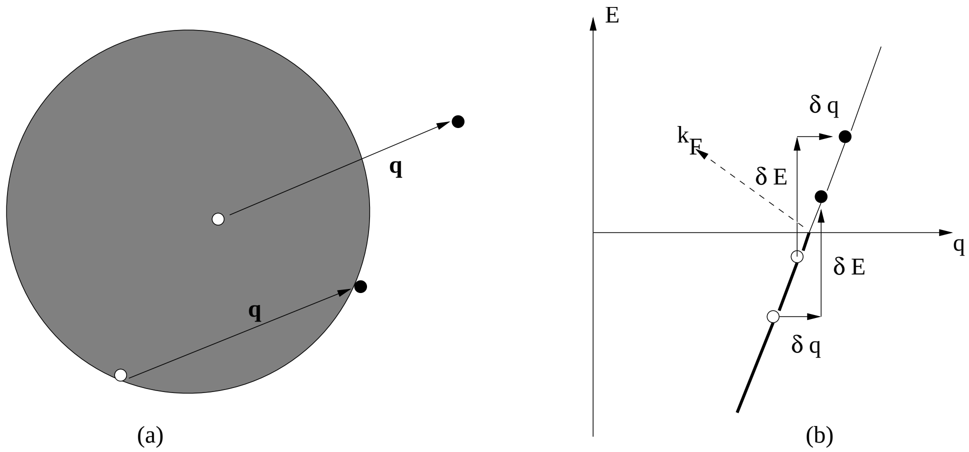

Figure 2: (a)For a fixed **q** in 2D, a particle-hole pair can have both a high and low energy of excitation. (b) In 1D, the excitation energy has a single value for a given δq.

▷ How does this peculiarity affect the physics?

[4]Lecture notes prepared by Vikram Tripathi

Interactions are usually expressed in density: $V(x-x')\hat{\rho}(x)\hat{\rho}(x')$. In second quantised form this looks like

$$\sum_{x,x'} V(x-x')\hat{\rho}(x)\hat{\rho}(x') = \sum_{k,p,q} V(q) c^\dagger_{k+q} c^\dagger_{p-q} c_k c_p \equiv \sum_q V(q)\hat{\rho}_q \hat{\rho}_{-q}, \tag{4}$$

where

$$\hat{\rho}_q = \sum_k c^\dagger_{k+q} c_q \tag{5}$$

is a charge density wave excitation with momentum q.

So if the particle-hole excitation $\rho(q)$ has a well-defined kinetic energy, then it means that $\rho(q)$ is an eigenstate:

$$\hat{H}\hat{\rho}(q)|0\rangle = E_q \hat{\rho}(q)|0\rangle. \tag{6}$$

• The statement is actually stronger. According to Bloch, the low energy states of a 1D electron system are completely exhausted by its density excitations.

▷ The Luttinger model

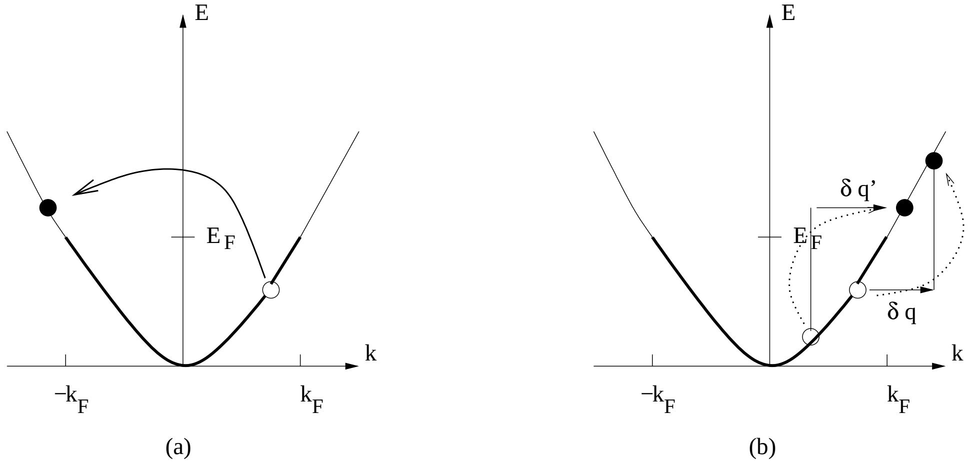

Figure 3: Violation of one-to-one correspondence. (a) Excitations involving momentum change of the order of $2k_F$ (backscattering). (b) Particle-hole excitations with large energy.

An idealised model of 1D electrons can be written down which excludes processes that violate this one-to-one correspondence of ρ_q and eigenstates of $\hat{H}$, and renders the model exactly solvable even with strong interactions, is called the Luttinger model. Haldane (1981) conjectured that all interacting 1D metals are *adiabalically continuable* from the Luttinger model leading to the concept of a Luttinger liquid. In the Luttinger model we

(a) Linearise the spectrum.

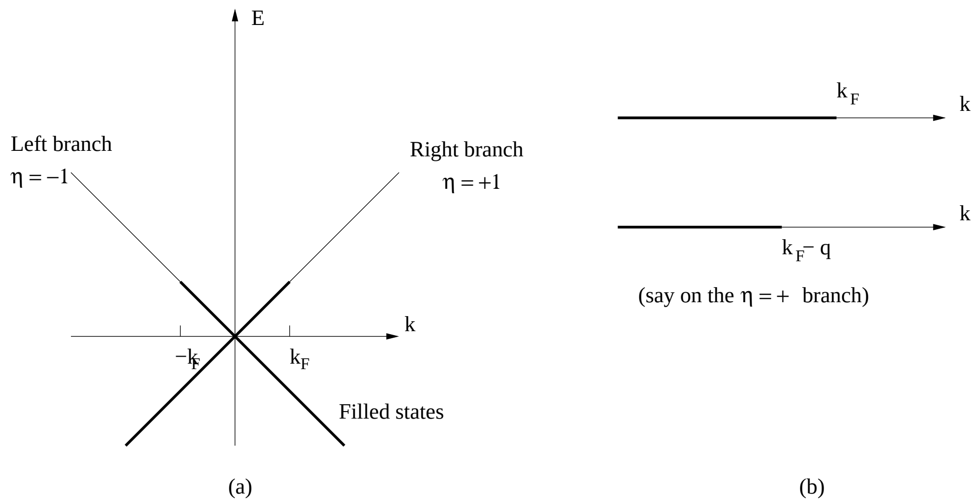

Figure 4: The Luttinger model. (a) Linearizing and adding an infinity of filled states (bold lines) with negative energy. (b) Illustration of n_{k-q} and n_q. Their difference appears in the commutation relation of densities $\rho_\eta(q)$.

(b) Add an infinity of filled states with negative energy.

Ignoring spin for the moment—write the density as

$$\rho_\eta(q) = \sum_k c^\dagger_{\eta,k+q} c_{\eta,k}.$$

This operator behaves like a boson:

$$[\rho_\eta(-q), \rho_{\eta'}(q')] = \delta_{\eta\eta'}\delta_{qq'}\frac{\eta q L}{2\pi}.$$

By simply substituting the expression for $\rho_\eta(q)$ in terms of electrons, the commutator gives

$$\delta_{\eta\eta'}\delta_{qq'}\sum_k (n_{k-q} - n_k).$$

For a finite bandwidth the $\sum_k$ gives zero (just put $\tilde{k} = k - q$) but with an infinite number of filled states we have

$$\sum_k (n_{k-q} - n_k) = \frac{\eta q L}{2\pi}.$$

This is how it works in the ground state but it just relies on there existing some k_0 below which all states are filled (in the ground state $k_0 = k_F$) so it is quite generally true.

We make a set of bosons now:

$$\begin{aligned} b^\dagger_{+,q} &= \sqrt{\frac{2\pi}{Lq}}\sum_k c^\dagger_{+,k+q} c_{+,k}; \\ b^\dagger_{-,q} &= \sqrt{\frac{2\pi}{Lq}}\sum_k c^\dagger_{-,k} c_{-,k+q}, \end{aligned} \tag{7}$$

so that the bosons $b_{\eta,q}$ satisfy the familiar Bose commutation relations,

$$[b_{\eta,q}, b^{\dagger}_{\eta' q'}] = \delta_{\eta\eta'}\delta qq'.$$

Since there is one-to-one corresponds between density fluctuations and the low-energy eigenstates of the Hamiltonian, we may express the original problem of electrons in terms of these bosons:

$$\begin{aligned} H_0 &= \frac{2\pi v_F}{L} \sum_{q>0,\eta=\pm 1} \rho_\eta(q)\rho_\eta(-q) \\ &= \sum_{q>0,\eta=\pm 1} v_F q\, b^{\dagger}_{\eta q} b_{\eta q}. \end{aligned} \tag{8}$$

• Introducing interactions ($g-$ology)

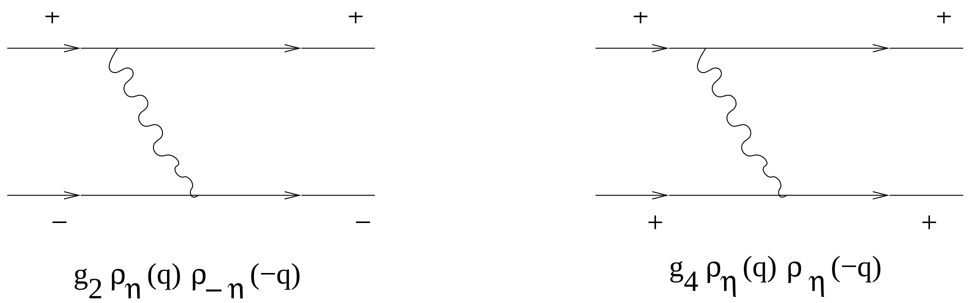

Figure 5: The $g-$ology of interactions. The diagram to the left shows interaction of left movers with right movers. The diagram to the right denotes interaction of the same species.

$$H = \sum_{q>0,\eta=\pm 1} v_F q\, b^{\dagger}_{\eta q} b_{\eta q} + \frac{1}{2L}\sum_{q>0,\eta=\pm 1} g_2 \rho_\eta(q)\rho_{-\eta}(-q) + \frac{1}{2L}\sum_{q>0,\eta=\pm 1} g_4 \rho_\eta(q)\rho_\eta(-q).$$

Now express this in terms of the bosons:

$$H = \sum_{q>0,\eta=\pm 1}\left[v_F q\, b^{\dagger}_{\eta q} b_{\eta q} + \frac{q g_4}{2\pi} b^{\dagger}_{\eta q} b_{\eta q} + \frac{q g_2}{4\pi}\left(b^{\dagger}_{\eta q} b^{\dagger}_{-\eta q} + b_{\eta q} b_{-\eta q}\right)\right].$$

The Hamiltonian is quadratic and so can be solved exactly via a Bogoliubov transformation (i.e., a rotation that permits mixing of b and $b^\dagger$) to a new basis of bosons α, $\alpha^\dagger$:

$$H = \sum_{q>0,\eta=\pm 1} q \cdot \sqrt{\left(v_F + \frac{g_4}{2\pi}\right)^2 - \left(\frac{g_2}{2\pi}\right)^2}\left[\alpha^{\dagger}_{\eta q}\alpha_{\eta q} + \frac{1}{2}\right]$$

So the interacting 1D electron model looks exactly like the Hamiltonian for waves on a stretched string with wave velocity $v = \sqrt{\left(v_F + \frac{g_4}{2\pi}\right)^2 - \left(\frac{g_2}{2\pi}\right)^2}$.

▷ Can we represent single electrons in terms of bosons?

To be really useful, we would like to know how to write a single electron in terms of the bosons ⇒ then we can use our solution to write correlation functions.

It turns out that you can do this (Luther and Peschel'74 and Haldane'81). Details will not be provided here. Refer to the review by Schulz I have cited in the reading list at the beginning of this lecture. One defines field operators (essentially going backwards in our second quantization treatment of the string) which are usually expressed in terms of $\rho_\pm$ or b, $b^\dagger$ and not the Bogoliubov fields α, $\alpha^\dagger$. This means the electron is truly lost in the many-body physics contained in the Bogoliubov fields.

Define the following two Bose fields in real space corresponding to displacement and the canonical momentum:

$$\phi(x) = -\frac{i\pi}{L}\sum_{q\neq 0}\frac{1}{q}e^{-\delta|q|/2-iqx}\left[\rho_+(q)+\rho_-(q)\right] - \frac{N\pi x}{L},$$

$$\Pi(x) = \frac{1}{L}\sum_{q\neq 0}e^{-\delta|q|/2-iqx}\left[\rho_+(q)-\rho_-(q)\right] + \frac{J}{N},$$

where $N = N_+ + N_-$ and $J = N_+ - N_-$ and δ is a small number to ensure convergence.

You may verify that the canonical commutation relations are satisfied:

$$[\phi(x), \Pi(x')] = i\delta(x-x').$$

• Comparison with a stretched string

Compare the Hamiltonian

$$H = \int dx\, \left[\frac{\pi u K}{2}\Pi(x)^2 + \frac{u}{2\pi K}(\nabla\phi)^2\right],$$

with a stretched string: $H = \int dx\, \left[\frac{1}{2\rho}\Pi(x)^2 + \frac{1}{2T}(\nabla\phi)^2\right]$. Here

$$u = \sqrt{\left(v_F + \frac{g_4}{2\pi}\right)^2 - \left(\frac{g_2}{2\pi}\right)^2}; \qquad K = \sqrt{\frac{2\pi v_F + g_4 - g_2}{2\pi v_F + g_4 + g_2}}.$$

• Expression for electron operator

$$\psi_\pm(x) = \lim_{\delta\to 0}\frac{1}{\sqrt{2\pi\delta}}U_\pm \exp\left[\pm ik_F x \mp i\phi(x) + i\pi\int^x \Pi(x')dx'\right].$$

▷ Including spin: spin-charge separation

So far we have just considered spinless fermions in 1D. Adding spin enables us to see that the spin and charge of an electron completely separate. There is no way to adiabatically continue from non-interacting electrons to the resulting excitations. Spin-charge separation occurs without explicitly spin dependent forces.

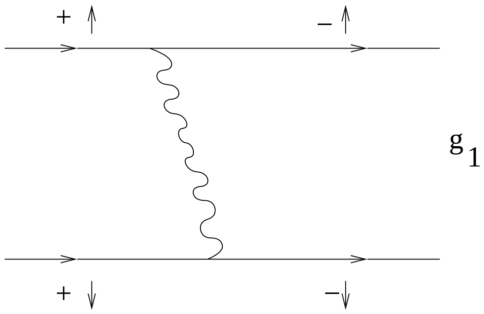

Figure 6: New kind of interaction involving spin. An interaction on the left branch scatters from one on the right branch and a spin flip occurs. This term cannot be expressed as a pair of bosons.

Adding spin to the electrons adds an additional label σ. It also allows a new type of interaction [See Fig.6] This term cannot be expressed as a pair of bosons, so I will ignore it for the moment.

$$H = \sum_{q>0,\sigma,\sigma'\eta=\pm1} \left[v_F q \delta_{\sigma\sigma'} b^\dagger_{\eta q\sigma} b_{\eta q\sigma} + \frac{q g_2}{4\pi} \left(b^\dagger_{\eta q\sigma} b^\dagger_{-\eta q\sigma'} + b_{\eta q\sigma} b_{-\eta q\sigma'} \right) + \frac{q g_4}{2\pi} b^\dagger_{\eta q\sigma} b_{\eta q\sigma'} \right]$$

Now writing

$$\begin{aligned} b^\dagger_{\eta qc} &= \frac{1}{\sqrt{2}} \left(b^\dagger_{\eta q\uparrow} + b^\dagger_{\eta q\downarrow} \right) \qquad \text{c: charge} \\ b^\dagger_{\eta qs} &= \frac{1}{\sqrt{2}} \left(b^\dagger_{\eta q\uparrow} - b^\dagger_{\eta q\downarrow} \right) \qquad \text{s: spin} \end{aligned}$$

Then we have

$$H = \sum_{q>0\,\eta=\pm1} \left[v_F q \left(b^\dagger_{\eta qc} b_{\eta qc} + b^\dagger_{\eta qs} b_{\eta qs} \right) + \frac{q g_2}{4\pi} \left(b^\dagger_{\eta qc} b^\dagger_{-\eta qc} + b_{\eta qc} b_{-\eta qc} \right) + \frac{q g_4}{2\pi} b^\dagger_{\eta qc} b_{\eta qc} \right],$$

which can again be diagonalised by a Bogoliubov transformation:

$$H = \sum_{q>0,\eta=\pm1} \left[q \cdot \sqrt{\left(v_F + \frac{g_4}{2\pi} \right)^2 - \left(\frac{g_2}{2\pi} \right)^2} \left(\alpha^\dagger_{\eta qc} \alpha_{\eta qc} + \frac{1}{2} \right) + q v_F \, \alpha^\dagger_{\eta qs} \alpha_{\eta qs} \right].$$

For the $q < 0$ branch, q should be replaced with $-q$ above.

- The charge and spin densities move with different velocities!

These excitations exhaust the Hilbert space of the Hamiltonian—this is the complete solution.

Spectral functions

Rather than write down the expression for the spinful electron, it's perhaps more useful to quote some results. Compare the spectral function $A(\omega)$ of a Fermi liquid and a Luttinger liquid:

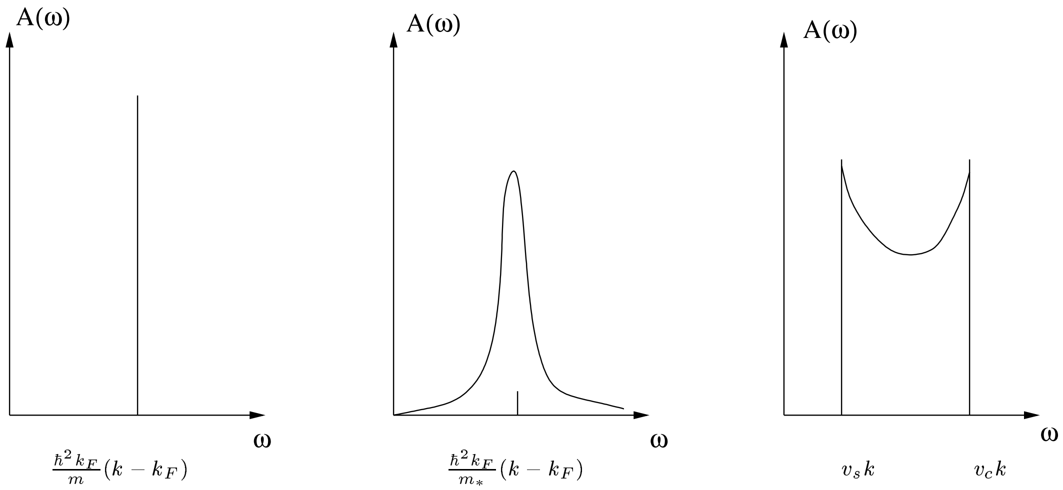

Figure 7: (Left to right) Spectral functions for non-interacting electrons, Fermi liquid and a Luttinger liquid respectively

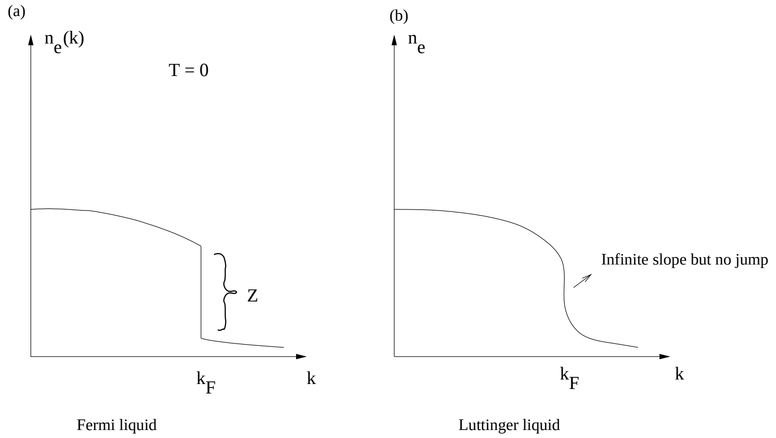

Figure 8: Electron distribution function in 1D for (a) a Fermi liquid and (b) a Luttinger liquid. In a Fermi liquid, the distribution function has a finite discontinuity at k_F while in a Luttinger liquid, the distribution function is continuous although its slope is infinite at k_F.

The $T = 0$ electron distribution function

▷ The Luttinger liquid hypothesis

You might wonder what happened to the possible g_1 interaction shown in Fig.6. Also, the electron spectrum is not quite linear which may lead to deviations from Luttinger liquid behaviour. Haldane (1981) suggested that in 1D, a new type of adiabatic continuity might operate: any interacting 1D metal is adiabatically continuable from the Luttinger model. (Formally, g_1 can be shown to be irrelevant). All of the physics is determined by three parameters: v_c (charge velocity or holon velocity), v_s (spin velocity or spinon velocity), and K_c, which controls power laws of the electron correlation function. The fourth possible parameter, $K_\sigma \equiv 1$ for physical interactions.

▷ Applications

- Quantum wires
- Organics: TTF–TCNQ (CDW instabilities)
- $SrCuO_2$ (observation of spinon and holon dispersion in photoemission experiments by Z.X.Shen *et al.*)