Lecture XVIII: Feynman Path Integral

▷ Motivation:

- Alternative formulation of QM (cf. canonical quantisation)
- Close to classical construction — i.e. semi-classics easily obtained
- Effective formulation of non-perturbative approaches
- Prototype of higher-dimensional field theories

▷ Time-dependent Schrödinger equation

$$\boxed{i\hbar\partial_t|\Psi\rangle = \hat{H}|\Psi\rangle}$$

Formal solution: $|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle = \sum_n e^{-iE_nt/\hbar}|n\rangle\langle n|\psi(0)\rangle$

▷ Time-evolution operator

$$|\Psi(t')\rangle = \hat{U}(t',t)|\Psi(t)\rangle, \qquad \hat{U}(t',t) = e^{-\frac{i}{\hbar}\hat{H}(t'-t)}\theta(t'-t)$$

N.B. Causal

- Real-space representation:

$$\Psi(q',t') \equiv \langle q'|\Psi(t')\rangle = \langle q'|\hat{U}(t',t)\int dq|q\rangle\langle q| \quad \wedge \quad |\Psi(t)\rangle = \int dq\, U(q',t';q,t)\Psi(q,t),$$

where $U(q',t';q,t) = \langle q'|e^{-\frac{i}{\hbar}\hat{H}(t'-t)}|q\rangle\theta(t'-t)$ — propagator or Green function

$$\left(i\hbar\partial_{t'} - \hat{H}\right)U(t'-t) = i\hbar\delta(t'-t)$$

N.B. $\partial_{t'}\theta(t'-t) = \delta(t'-t)$

Physically: $U(q',t';q,t)$ describes probability amplitude for particle to propagate from q at time t to q' at time t'

▷ Construction of Path Integral

Feynman's idea: divide time evolution into $N \to \infty$ discrete time steps $\Delta t = t/N$

$$e^{-i\hat{H}t/\hbar} = \left[e^{-i\hat{H}\Delta t/\hbar}\right]^N$$

Then separate the operator content so that momentum operators stand to the left and position operators to the right

$$e^{-i\hat{H}\Delta t/\hbar} = e^{-i\hat{T}\Delta t/\hbar}e^{-i\hat{V}\Delta t/\hbar} + O(\Delta t^2)$$

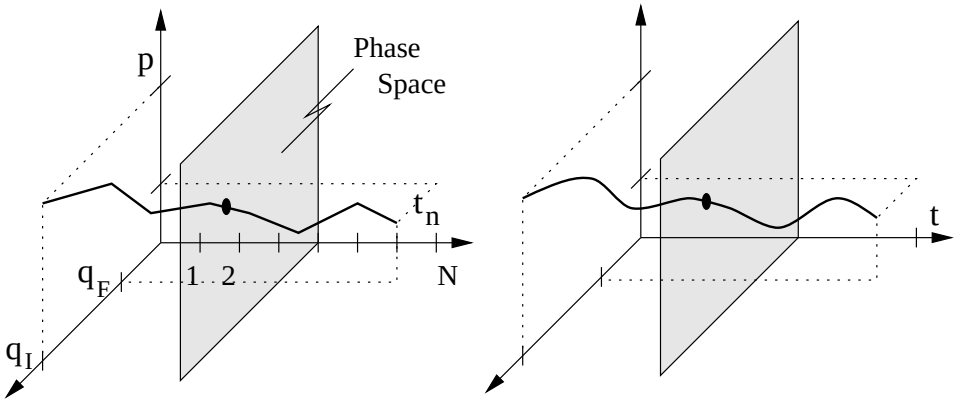
$$\langle q_F| \left[e^{-i\hat{H}\Delta t/\hbar}\right]^N |q_I\rangle$$
$$\simeq \langle q_F|_\wedge e^{-i\hat{T}\Delta t/\hbar}e^{-i\hat{V}\Delta t/\hbar}{}_\wedge e^{-i\hat{T}\Delta t/\hbar}e^{-i\hat{V}\Delta t/\hbar}{}_\wedge \cdots {}_\wedge e^{-i\hat{T}\Delta t/\hbar}e^{-i\hat{V}\Delta t/\hbar}|q_I\rangle$$

Insert at $\wedge$ resol. of id. $= \int dq_n \int dp_n |q_n\rangle\langle q_n|p_n\rangle\langle p_n|$, and using $\langle q|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{iqp/\hbar}$,

$$e^{-i\hat{V}\Delta t/\hbar}|q_n\rangle\langle q_n|p_n\rangle\langle p_n|e^{-i\hat{T}\Delta t/\hbar} = |q_n\rangle e^{-iV(q)\Delta t/\hbar}\langle q_n|p_n\rangle e^{-iT(p)\Delta t/\hbar}\langle p_n|,$$

and $$\langle p_{n+1}|q_n\rangle\langle q_n|p_n\rangle = \frac{1}{2\pi\hbar}\exp\left[iq_n(p_n - p_{n+1})\right]$$

$$\langle q_F|e^{-i\hat{H}t/\hbar}|q_I\rangle = \int_{q_N=q_F,q_0=q_I} \prod_{n=1}^{N-1} dq_n \prod_{n=1}^{N} \frac{dp_n}{2\pi\hbar} \exp\left[-\frac{i}{\hbar}\Delta t \sum_{n=0}^{N-1}\left(V(q_n) + T(p_{n+1}) - p_{n+1}\frac{q_{n+1}-q_n}{\Delta t}\right)\right]$$



i.e. at each time step, integration over the classical phase space coords. $x_n \equiv (q_n, p_n)$

Contributions from trajectories where $(q_{n+1} - q_n)p_{n+1} > \hbar$ are negligible

motivates continuum limit

$$\langle q_F|e^{-i\hat{H}t/\hbar}|q_I\rangle = \overbrace{\int_{q(t)=q_F,q(0)=q_I} D(q,p)}^{\int_{q_N=q_F,q_0=q_I}\prod_{n=1}^{N-1}dq_n\prod_{n=1}^{N}\frac{dp_n}{2\pi\hbar}} \exp\Big[-\frac{i}{\hbar}\overbrace{\int_0^t dt'}^{\Delta t\sum_{n=0}^{N-1}}\Big(\overbrace{V(q_n)+T(p_{n+1})}^{H(\mathbf{x}|_{t'=t_n})} - \overbrace{p_{n+1}\frac{q_{n+1}-q_n}{\Delta t}}^{p\dot{q}|_{t'=t_n}}\Big)\Big]$$

Propagator expressed as functional integral:

Hamiltonian formulation of Feynman Path Integral

$$\langle q_F|e^{-i\hat{H}t/\hbar}|q_I\rangle = \int_{q(t)=q_F,q(0)=q_I} D(q,p)\exp\Big[\frac{i}{\hbar}\overbrace{\int_0^t dt'\overbrace{(p\dot{q}-H(p,q))}^{\text{Lagrangian}}}^{\text{Action}}\Big]$$

Quantum transition amplitude expressed as sum over all possible phase space trajectories (subject to appropriate b.c.s) and weighted by classical action

▷ Lagrangian formulation: for "free-particle" Hamiltonian $H(p,q) = p^2/2m + V(q)$

$$\langle q_F|e^{-i\hat{H}t/\hbar}|q_I\rangle = \int_{q(t)=q_F,q(0)=q_I} Dq e^{-(i/\hbar)\int_0^t dt' V(q)} \int Dp \overbrace{\exp\left[-\frac{i}{\hbar}\int_0^t dt'\left(\frac{p^2}{2m} - p\dot{q}\right)\right]}^{\text{Gaussian integral on p}}$$

$$\frac{p^2}{2m} - p\dot{q} \mapsto \frac{1}{2m}\overbrace{(p - m\dot{q})^2}^{p'^2} - \frac{1}{2}m\dot{q}^2$$

Functional integral justified by discretisation

$$\boxed{\langle q_F|e^{-i\hat{H}t/\hbar}|q_I\rangle = \int_{q(t)=q_F,q(0)=q_I} Dq \exp\left[\frac{i}{\hbar}\int_0^t dt'\left(\frac{m\dot{q}^2}{2} - V(q)\right)\right]}$$

$$Dq \to \widetilde{D}q = \lim_{N\to\infty}\left(\frac{Nm}{it2\pi\hbar}\right)^{N/2}\prod_{n=1}^{N-1} dq_n$$